Methods used to calculate efficiency and work for Crowley's method of isothermal compression and expansion

Introduction

This paper defines what is meant by the term "near isothermal process" in terms of an isothermal efficiency (z). It then shows how the isothermal efficiency (z) can be used to calculate the work done in a polytropic process and how the isothermal efficiency can be calculated knowing the geometry of the heat absorbing and releasing structure (HARS).

This paper assumes the reader is familiar with Crowley's method of isothermal compression and expansion.

Definition and Formulas

Isothermal efficiency is sometimes defined by others as:

$$\zeta_{iso} = \frac{Isothermal\ work}{Actual\ work}$$

We prefer to use a different definition of isothermal efficiency, using temperature change to calculate were a process lays between being fully adiabatic and fully isothermal. So if there is no temperature change the process is fully isothermal and has an efficiency of 1, but if the temperature change is the same as an adiabatic process then the efficiency is 0. We define the isothermal efficiency as.

$$z = \frac{\Delta T_\gamma - \Delta T_n}{\Delta T_\gamma} \qquad (1)$$

Where

z   Isothermal Efficiency

$\Delta T\Upsilon$ Temperature change for an adiabatic process index ϒ=Cp/Cv

ΔTn Temperature change for actual process with polytropic index n

It will also be shown below, that

$$z = \frac{P_\gamma - P_n}{P_\gamma - P_I} \qquad (2)$$

Where

$P\Upsilon$ Pressure after an adiabatic process

PI Pressure after an isothermal process

Pn Pressure after actual process with polytropic index n

Using definition of efficiency (z) in equations (1) or (2) allows a direct calculation of the work in a polytropic process. It will be proved below that the actual work

WDn = WDI + (1-z) (WDϒ-WDI)   (3)

Where

WDn   Work done for polytropic process with index n

WDI   Work done for isothermal process

WDϒ   Work done for adiabatic process with index of ϒ

From equation (3) it can be seen that when the isothermal efficiency z is 1 the work done for a given volume change is the isothermal work done. The actual work changes linearly with reducing isothermal efficiency (z) such that when z equals 0 the work is the adiabatic work done.

For a HARS similar to the type shown in figure 1 and used in an isothermal process it will also be shown that approximately

$$z = 1 - 0.81e^{4.93(1/\gamma - 1)K.Nu.T/(Pn.Hz.G^2)} \qquad (4)$$

When the Fourier number $\frac{2(1-\frac{1}{\gamma})NuKTn}{Pn.Hz.G^2} > 0.2$ (5)

Where

K   gas thermal conductivity

Nu   Nusselt Number

T   Gas temperature (isothermal)

Pn   Maximum Gas Pressure (isothermal)

Hz   Speed of compressor in hertz

G   Gap between sheets of HARS

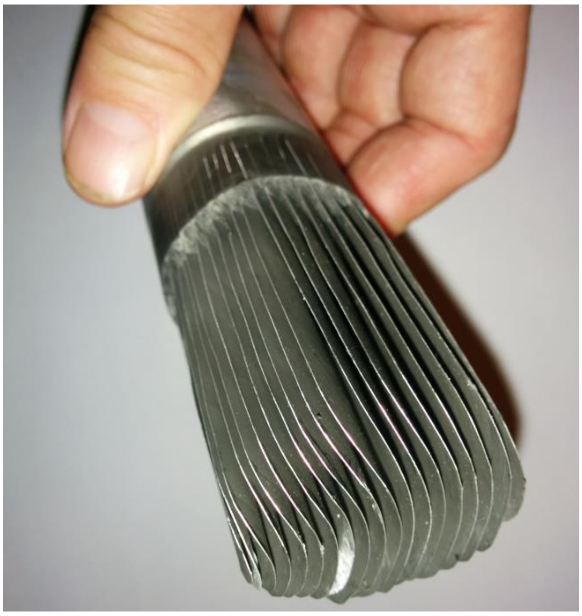

Figure 1

Proof of Equation (2)

Equation (1) is a definition so does not require a proof.

Equation (2) is derived from equation (1) as follows.

For a polytropic process let

V1	Initial volume
V	Actual volume
n	Polytropic index
γ	Adiabatic index Cp/Cv
T1	Initial temperature
Tn	Temperature for polytropic process at volume V
TI=T1	Temperature for isothermal process at volume V
$T\Upsilon$	Temperature for adiabatic process at volume V
P1	Initial pressure
Pn	Pressure at volume V for polytropic process
$P\Upsilon$	Pressure for adiabatic process at volume V
PI	Pressure for isothermal process at volume V

$$P\gamma = P1.\left(\frac{V1}{V}\right)^{\gamma} \qquad (7)$$

$$Pn = P1.\left(\frac{V1}{V}\right)^{n} \qquad (8)$$

$$PI = \frac{P1.V1}{V} \qquad (9)$$

Dividing equation 7 by 9

$$\frac{P\gamma}{PI} = \left(\frac{V1}{V}\right)^{\gamma}\frac{V}{V1} = \left(\frac{V1}{V}\right)^{(\gamma-1)}$$

So $$\left(\frac{V1}{V}\right)^{(\gamma-1)} = \frac{P\gamma}{PI} \qquad (10)$$

Similarly $$\left(\frac{V1}{V}\right)^{(n-1)} = \frac{Pn}{PI} \qquad (11)$$

$$\frac{P1.V1}{T1} = \frac{Pn.V}{Tn}$$

So $$Tn = \frac{T1.Pn.V}{P1.V1} \qquad (12)$$

Similarly $$T\gamma = \frac{T1.P\gamma.V}{P1.V1} \qquad (13)$$

Substituting equation (8) for Pn into equation (12)

$$Tn = T1.P1\left(\frac{V1}{V}\right)^{n}.\frac{V}{P1.V1}$$

Therefore $$Tn = T1.\left(\frac{V1}{V}\right)^{(n-1)} \qquad (14)$$

Similarly $$T\Upsilon = T1.\left(\frac{V1}{V}\right)^{(\gamma-1)} \qquad (15)$$

$$\Delta T\Upsilon = T\Upsilon - \mathrm{T1} = \mathrm{T1}.\left(\frac{V1}{V}\right)^{(\gamma-1)} - T1 \qquad (16)$$

$$\Delta Tn = Tn - \mathrm{T1} = \mathrm{T1}.\left(\frac{V1}{V}\right)^{(n-1)} - T1 \qquad (17)$$

Substituting for $\Delta T\Upsilon$ & ΔTn into equation (1)

$$z = \frac{\mathrm{T1}.\left(\frac{V1}{V}\right)^{(\gamma-1)} - \mathrm{T1}.\left(\frac{V1}{V}\right)^{(n-1)}}{\mathrm{T1}.\left(\frac{V1}{V}\right)^{(n-1)} - T1}$$

Dividing through by T1

$$z = \frac{\left(\frac{V1}{V}\right)^{(\gamma-1)} - \left(\frac{V1}{V}\right)^{(n-1)}}{\left(\frac{V1}{V}\right)^{(n-1)} - 1} \qquad (18)$$

Substituting equations (10) & (11) into equation (18)

$$z = \frac{\frac{P\gamma}{PI} - \frac{Pn}{PI}}{\frac{P\gamma}{PI} - 1} \qquad (19)$$

Multiplying equation (19) through by PI

$$z = \frac{P\gamma - Pn}{P\gamma - PI} \qquad (2)$$

So $z = \frac{\Delta T_\gamma - \Delta T_n}{\Delta T_\gamma} = \frac{P\gamma - Pn}{P\gamma - PI}$

Proof of Equation (3)

Rearranging equation (2)

$$Pn = P\gamma - z(P\gamma - PI)$$

$$Pn = (1-z)P\gamma + z.PI \qquad (20)$$

Let x=1-z

So $Pn = x.P\gamma + z.PI$ (21)

Using equation (7) and (9) to substitute for PY and PI in equation (21)

$$Pn = x.P1.\left(\frac{V1}{V}\right)^{\gamma} + z.\frac{P1.V1}{V} \qquad (22)$$

$$Pn = z.P1.V1.V^{-1} + x.P1.V1^{\gamma}.V^{-\gamma} \qquad (23)$$

Now $WDn = \int_{V1}^{V} Pn\, dV$

$$= [^{V}_{V1} \mathrm{z.P1.V1.ln(V)} + x.\frac{P1.V1^{\gamma}}{1-\gamma}V^{1-\gamma}]$$

$$= zP1.V1ln\left(\frac{V}{V1}\right) + x.\frac{P1.V1^{\gamma}}{1-\gamma}[V^{1-\gamma} - V1^{1-\gamma}]$$

$$= zP1.V1ln\left(\frac{V}{V1}\right) + \frac{x}{1-\gamma}\left[P1.\left(\frac{V1}{V}\right)^{\gamma} V - P1.V1\right]$$

But $\left(\frac{V1}{V}\right)^{\gamma} = \frac{P\gamma}{P1}$

So

$$WDn = zP1.V1ln\left(\frac{V}{V1}\right) + \frac{X}{1-\gamma}[P\gamma.V - P1.V1] \qquad (24)$$

But $WDI = P1.V1ln\left(\frac{V}{V1}\right)$ (25)

And $WD\gamma = \frac{1}{1-\gamma}[P\gamma.V - P1.V1]$ (26)

Formulas (25) and (26) for the work done isothermally and adiabatically are taken from “Fluid Mechanics” 4th edition page 21 by Douglas, Gasiorek and Swaffield

So WDn = z.WDI + x.WDϒ

But x = 1-z

WDn = z.WDI + (1-z).WDY

Rearranging we get

WDn = WDI + (1-z) (WDϒ-WDI) (3)

Proof of Equation (4) and (5)

This proof is based on methods defined in “Heat transfer a Practical Approach” by Yunus A Cengel. 1998. Pages 232 to 235.

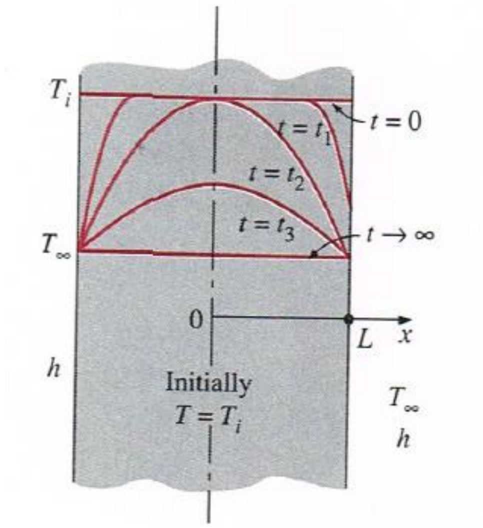

Figure 2

Cengel method uses 4 dimensionless numbers to define a temperature profile with time and position inside a large plane wall as shown in figure 2. At time t=0 the initial temperature is Ti, and as t→∞ the temperature is T∞. The Cengel method allows for the general heat transfer case where at the boundary wall at distance L from the centre there is an insulating layer with heat transfer coefficient h, so the temperature at L is higher than T∞. However in the case of the HARS used in Crowley’s method of compression and expansion the boundary layer are the sheets of the HARS which are practically at a constant temperature and it is only the gas between the sheets of the HARS which changes temperature during a compression or expansion cycle.

If there was no HARS the temperature of the gas would change adiabatically and this temperature would be constant through the whole volume of the gas. But the compression or expansion of the gas occurs over a fixed period of time (half the piston cycle time). During this half cycle period, heat is transferred between the gas and HARS and it is this heat transfer that ensures the process is near isothermal. Using the method defined by Cengel it is possible to calculate the dimensionless temperature profile between the sheets of the HARS at the end of the compression or expansion cycle and from this temperature profile the efficiency (z) can be calculated.

Dimensionless temperature as defined in Cengel-

$$\theta_{(x,t)} = \frac{T_{(x,t)} - T_\infty}{T_i - T_\infty} \qquad (27)$$

To calculate the efficiency we need to know the average temperature of the gas.

At some distance "a" from the centre line the actual gas temperature will be the same as the average gas temperature. And it is this average temperature we use to calculate the efficiency (z).

Average dimensionless temp $\theta_{(a,t)}=\frac{T_{(a,t)}-T_\infty}{T_i-T_\infty}$ (27)

Now $\frac{T_i-T_\infty}{T_i-T_\infty}=1$

So $1-\ \theta_{(a,t)}=\frac{T_i-T_\infty}{T_i-T_\infty}-\frac{T_{(a,t)}-T_\infty}{T_i-T_\infty}$

$$1-\ \theta_{(a,t)}=\frac{(T_i-T_\infty)-(T_{(a,t)}-T_\infty)}{T_i-T_\infty}$$

If initial temperature (T_i) is due to an adiabatic compression. From inspection of figure 2 it can be seen that

$$\Delta T\Upsilon=T_i-T_\infty$$

And $\Delta T\mathrm{n}=T_{(a,t)-T_\infty}$

So $1-\ \theta_{(a,t)}=\frac{\Delta T\Upsilon-\Delta T\mathrm{n}}{\Delta T\Upsilon}$

So from equation 1

$$z=1-\ \theta_{(a,t)} \quad (28)$$

Biot number $Bi=\frac{hL}{k}$ (29)

In this case temperature at the edges in figure 2 is at a constant temperature because the edges are the sheets of the HARS so h=∞

Therefore Bi=∞ (30)

From Cengel for parallel walls

$$Q_{(x,t)}=A_1e^{-\lambda_1^2\tau}.\cos(\lambda_1\,x/L) \quad (31)$$

The Heisler chats (table 4-1 Cengel) gives the following values when Biot number Bi=∞

$$A_1=1.2732 \quad (32)$$

$$\lambda_1=1.5708=\frac{\pi}{2} \quad (33)$$

The term $\cos(\lambda_1\,x/L)$ in equation (31) generates the temperature profile from the centre to the edge when x=L. To find the average dimensionless temperature the average value of this term is required.

$$\text{Average value}=\frac{1}{L}\int_0^L cos\left(\frac{\pi}{2}\frac{x}{L}\right)\,dx$$

$$=\frac{1}{L}\left[{}_0^L\frac{2L}{\pi}Sin\left(\frac{\pi}{2}\frac{x}{L}\right)\right]$$

$$=\frac{2}{\pi} \quad (34)$$

Substituting (32), (33) & (34) into (31)

$$Q_{(a,t)}=0.8105e^{-2.467\tau} \quad (35)$$

Substituting (35) into (28)

$$z=1-0.8105e^{-2.467\tau} \quad (36)$$

Dimensionless time

Fourier number $\tau=\frac{\alpha t}{L^2}$ (37)

Thermal diffusivity $\alpha=\frac{Nu.K}{\rho.C_p}$ (38)

Density $\rho=\frac{Pn}{RTn}$ (39)

But R=Cp-Cv

So $\rho=\frac{Pn}{(Cp-Cv)Tn}$ (40)

Substituting (40) into (38)

$$\alpha=\frac{(Cp-Cv)Nu.K.Tn}{Cp.Pn}$$

$$\alpha=\left(1-\frac{1}{\gamma}\right)\frac{Nu.K.Tn}{Pn} \quad (41)$$

The time for heat transfer $t=\frac{1}{2Hz}$ (42)

Distance L=$\frac{G}{2}$ (43)

Substituting (41), (42) and (43) into (37)

$$\tau=\left(1-\frac{1}{\gamma}\right)\frac{2Nu.K.Tn.}{Pn.Hz.G^2} \quad (44)$$

Substituting equation (44) into (36)

$$z=1-0.81e^{4.93(1/\gamma-1)K.Nu.T/(Pn.Hz.G^2)} \quad (4)$$

Cengel states that equation 31 is only valid for Fourier number greater than 0.2

So $\tau>0.2$ (45)

Substituting (44) into (45)

When Fourier number $\frac{2\left(1-\frac{1}{\gamma}\right)Nu.KTn}{Pn.Hz.G^2}>0.2$ (5)

When proving equation (4) a number of implied assumptions were made. Some of these assumptions may tend to overestimate the efficiency (z) while others will tend to underestimate the efficiency. The main assumptions are discussed below.

The main implied assumption is that the heat to be transferred in or out of the HARS is available at the start of the cycle, however the heat is generated during the compression or expansion stroke so the time available for heat transfer will be less than that implied by equation (42).

However the piston will usually be driven by a crank so as the piston approaches the end of the stroke its velocity will reduce allowing proportionally more time for the heat to be transferred at the end of the stroke.

The thermal diffusivity equation (41) assumes a constant (maximum) pressure Pn which will result in the minimum diffusivity. However for most of the piston stroke the pressure will be a lot less so the diffusivity and hence isothermal efficiency (z) will be higher.

Further work is required to improve the accuracy of equation (4) by accounting for these assumptions in the model.

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